Tentamen Statistische Fysica **26 januari 2012**

Write your name and student number on each sheet that you hand in.

Problem 1

A spin system has a three-fold degenerate ground state and a two-fold degenerate excited state at energy ε. The interaction between the spins can be neglected. The total number of spins is indicated by N. The number of spins in the excited state is indicated by n.

a) Make plausible that the number of microstates of the system can written as

$$\Omega(n) = \frac{N!\, 3^{N-n}\, 2^{n}}{(N-n)!\, n!}$$

b) Calculate the energy of this system (in terms of ε, k, T and N) using the microcanonical ensemble method.

c) Calculate the heat capacity of the system.

Problem 2

A diatomic molecule may to a good approximation be considered as vibrating in one-dimensional simple harmonic motion with circular frequency ω. According to quantum mechanics such a system possesses an infinite set of non degenerate energy levels with energies $\varepsilon = \hbar\omega\left(r + \frac{1}{2}\right)$, r = 0, 1, 2, 3, ...

a) Show that the vibrational partition function is given by

$$Z_{vib} = \frac{e^{-\frac{\beta\hbar\omega}{2}}}{1 - e^{-\beta\hbar\omega}}, \text{ where } \beta = \frac{1}{kT}.$$

b) Derive an expression for the average vibrational energy of a molecule.

c) How does the vibrational partition function Z_{vib}, combine with the rotational partition function Z_{rot} and translational partition function Z_{tr} of the molecule to the total partition function Z of the entire gas (with N particles).

d) Derive an expression for the vibrational contribution to the entropy of the gas.

Problem 3

a) Draw a p-T phase diagram for a normal substance, indicating solid, liquid and gaseous phases as well as the special points in the diagram.

b) The vapour pressure curve is given by $P = C\,\mathbf{exp}\left\{-\frac{\Delta_{vap}H}{RT}\right\}$

In this expression $\Delta_{vap}H$ denotes the latent heat of vaporization per mol; R is the gas constant.

Give a derivation of this formula starting from the equation of Clausius-Clapeyron. Determine the constant C in terms of given quantities and the normal boiling point T_b at pressure P_0 (= 1 atm.)

Problem 4

The partition function of a gas can be written as

$Z = \sum_{\{n_1, n_2, n_3 .....\}} e^{-\beta\{n_1\varepsilon_1 + n_2\varepsilon_2 + n_3\varepsilon_3 + ...\}}$ where the sum goes over all the possible values of the occupation numbers n_i of the single particle states.

a) Derive, starting from the expression given above, the following relation for the average occupation number of the single particle state with energy ε_i :

$$\bar{n}_i = -\frac{1}{\beta}\frac{\partial \ln Z}{\partial \varepsilon_i}$$

b) Show that the partition function of a photon gas is given by $Z_{ph} = \prod_{r=1}^{\infty} \frac{1}{1 - e^{-\beta\varepsilon_r}}$

c) Show that the average number of photons of frequency ω is given by

$$\bar{n}(\omega) = \frac{1}{e^{\beta\hbar\omega} - 1}$$

d) Show that the total number of photons N is given by $N = b\frac{Vk^3T^3}{\hbar^3\pi^2c^3}$

Hint: Remember that the density of states for a particles in a box with volume V is given by $f(p)dp = \frac{4\pi V p^2 dp}{h^3}$.

Problem 5

Consider a gas of N non interacting purely relativistic electrons with single particle energy $\varepsilon = pc$. V denotes the volume of the gas.

a) Make plausible that the density of states for a free electron gas is given by

$$f(p) = \frac{8\pi V p^2 dp}{h^3}$$

The average occupation number of a single particle state with energy ε is given by:

$$n(\varepsilon) = \frac{1}{e^{\beta(\varepsilon-\mu)} + 1}$$

b) Derive an expression for the Fermi energy ε_F of this gas (in terms of N, V, h and c).

c) Calculate the total energy of the gas assuming T = 0 (in terms of N and ε_F).

d) Derive an expression for the pressure of this gas assuming T =0 (in terms of N, V, h and c).

Physical constants:

Getal van Avogadro: $N_A = 6{,}02 \times 10^{23}\ \text{mol}^{-1}$

Constante van Planck: $h = 6{,}626 \times 10^{-34}$ Js

$\hbar = \frac{h}{2\pi} = 1{,}055 \times 10^{-34}$ Js

Constante van Boltzmann: $k = 1{,}381 \times 10^{-23}\ \text{J K}^{-1}$

Gasconstante: $R = 8{,}315\ \text{J mol}^{-1}\ \text{K}^{-1}$

Lichtsnelheid: $c = 3 \times 10^{8}\ \text{m s}^{-1}$

Rustmassa elektron $m_e = 9{,}11 \times 10^{-31}$ kg

Rustmassa proton $m_p = 1{,}67 \times 10^{-27}$ kg

Bohr magneton $\mu_B = 9{,}27 \times 10^{-24}\ \text{A m}^2$

Integrals:

n	$\int_0^\infty dx\, x^n e^{-ax^2} \quad (a > 0)$	$\int_0^\infty \frac{x^n dx}{e^x - 1}$	$\int_0^\infty \frac{x^n dx}{e^x + 1}$	$\int_0^\infty \frac{x^n e^x}{(e^x - 1)^2}$	$\int_0^\infty x^n \ln(1 - e^{-x}) dx$
0	$\frac{1}{2}\sqrt{\left(\frac{\pi}{a}\right)}$	diverges	ln 2	diverges	$-\frac{\pi^2}{6}$
1/2	$\frac{0{,}6127}{a^{3/4}}$	$2{,}612\ \frac{\sqrt{\pi}}{2}$	0,6781	diverges	$-1{,}341\ \frac{\sqrt{\pi}}{2}$
1	$\frac{1}{2a}$	$\frac{\pi^2}{6}$	$\frac{\pi^2}{12}$	diverges	-1,202
3/2	$\frac{0{,}4532}{a^{5/4}}$	$1{,}341\ \frac{3\sqrt{\pi}}{4}$	1,153		$-1{,}127\ \frac{3\sqrt{\pi}}{4}$
2	$\frac{1}{4a}\sqrt{\frac{\pi}{a}}$	2,404	1,803	$\frac{\pi^2}{3}$	$-\frac{\pi^4}{45}$
5/2	$\frac{1{,}662}{a^{7/4}}$	$1{,}127\ \frac{15\sqrt{\pi}}{8}$	3,083		-3,505
3	$\frac{1}{2a^2}$	$\frac{\pi^4}{15}$	$\frac{7\pi^4}{120}$	7,212	-6,221
7/2	$\frac{0{,}5665}{a^{9/4}}$	12,268	11,184		
4	$\frac{3\sqrt{\pi}}{8a^{5/2}}$	24,886	23,331	$\frac{4\pi^4}{15}$	